A NOTE ON LOCAL GRADIENT ESTIMATE ON NEGATIVELY CURVED ALEXANDROV SPACES

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Abstract. In this note, we extend the local gradient estimate, Yau's gradient estimate, for harmonic functions by Zhang-Zhu [23] on negatively curved Alexandrov spaces to the one analogous to the Riemannian case.

1. Introduction

In 1975, Yau [20] proved that complete Riemannian manifolds with nonnegative Ricci curvature have the Liouville property. Later, Cheng-Yau [4] proved the following local version of Yau's gradient estimate.

Theorem A (Yau [20], Cheng-Yau [4]). *Let M^n be an n-dimensional complete noncompact Riemannian manifold with Ricci curvature bounded from below by $-K$ $(K \geq 0)$. Then there exists a constant $C = C(n)$, depending only on n, such that every positive harmonic function u on geodesic ball $B_{2R} \subset M$ satisfies*

$$\frac{|\nabla u|}{u} \leq C\frac{1+\sqrt{K}R}{R} \quad \text{in } B_R.$$

The regularity of harmonic functions on Alexandrov spaces is a challenging problem because of the lack of the smoothness of the metrics. The Hölder continuity of harmonic functions is well-known, see e.g. Kuwae-Machigashira-Shioya [6]. In 1996, Petrunin [15] proved the Lipschitz continuity of harmonic functions on Alexandrov spaces. Recently, Zhang-Zhu [21] introduced a notion of Ricci curvature on Alexandrov spaces. Using a delicate argument initiated by Petrunin, Zhang-Zhu [23] proved the Bochner formula on Alexandrov spaces which gives a quantitative estimate, i.e. Yau's gradient estimate for harmonic functions.

Theorem B (Zhang-Zhu [23]) *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ $(K \geq 0)$, and let Ω be a bounded domain in M. Then there exists a constant $C = C(n, \sqrt{K}\mathrm{diam}(\Omega))$, depending on n and $\sqrt{K}\mathrm{diam}(\Omega)$,*

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such that every positive harmonic function u on Ω satisfies

$$\frac{|\nabla u|}{u} \leq C\frac{1+\sqrt{K}R}{R} \quad \text{in } B_R. \tag{1}$$

for any geodesic ball $B_{2R} \subset \Omega$. If $K = 0$, the constant C depends only on n.

For the case $K > 0$, namely, Ricci curvature has a negative lower bound, Theorem B seems not satisfactory, compared with Theorem A, since the constant C depends not only on the dimension n but also on $\sqrt{K}R$. In this note, we refine the argument of Zhang-Zhu [23] for the negatively curved case from which we derive a local gradient estimate analogous to the Riemannian case. Our main result is in the following.

Theorem 1.1. *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exists a constant $C = C(n)$, depending only on n such that every positive harmonic function u on geodesic ball $B_{2R} \subset M$ satisfies*

$$\frac{|\nabla u|}{u} \leq C\frac{1+\sqrt{K}R}{R} \quad \text{in } B_R.$$

Due to the lack of regularity of harmonic functions on Alexandrov spaces, one cannot use the method of maximum principle as Yau [20] and Cheng-Yau [4]. As in Zhang-Zhu [23], we start with the Bochner formula established in [23] and use the Moser's iteration argument. The improvement is that we use a local uniform Sobolev inequality (see Theorem 3.1) with a refined constant for Alexandrov spaces with Ricci curvature bounded from below by $-K$ and get an L^β estimate of $|\nabla u|^2$ for $\beta \sim 1 + \sqrt{K}R$ which serves as the starting point of the Moser's iteration. This is adapted from the idea of Wang-Zhang [18]. The similar idea has been successfully applied to Finsler manifolds by the second author [19].

The paper is organized as follows: In the first section, we recall some basic and known results on Alexandrov spaces. In the next section, we prepare the analytic tool , i.e. Sobolev inequality, for Moser's iteration. The last section is devoted to the proof of Theorem 1.1.

2. Preliminaries on Alexandrov spaces

A metric space (X, d) is called an Alexandrov space if it is a complete locally compact geodesic space with sectional curvature bounded below locally in the sense of Alexandrov, i.e. locally satisfying Toponogov's triangle comparison, and of finite Hausdorff dimension. We refer to [1, 2] for the basic facts of Alexandrov spaces. It is well-known that the Bishop-Gromov volume comparison holds on Alexandrov spaces.

Kuwae-Shioya [7, 8, 9, 10] introduced and investigated a notion of infinitesimal Bishop-Gromov volume comparison, $BG(n, \kappa)$. Let (X, d) be an n-dimensional Alexandrov space.

Set for any $\kappa \in \mathbb{R}$,

$$\mathrm{sn}_\kappa(t) = \begin{cases} \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}, & \kappa > 0, \\ t, & \kappa = 0, \\ \frac{\sinh(\sqrt{|\kappa|}t)}{\sqrt{|\kappa|}}, & \kappa < 0. \end{cases}$$

For any $p \in X$, denote by $\mathrm{r}_p(x) := d(x, p)$ the distance function from p. For $p \in X$ and $0 < t \leq 1$, we define a subset $W_{p,t} \subset X$ and a map $\Phi_{p,t} : W_{p,t} \to X$ as follows: $x \in W_{p,t}$ if and only if there exists some $y \in X$ such that $x \in py$ and $\mathrm{r}_p(x) : \mathrm{r}_p(y) = t : 1$, where py is a minimal geodesic (shortest path) from p to y. For any $x \in W_{p,t}$, such y is unique (by Toponogov's triangle comparison) and we define $\Phi_{p,t}(x) = y$. Let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure on (X, d). The infinitesimal Bishop-Gromov condition for X with the measure $\mathcal{H}^n$, $BG(n, \kappa)$, is defined as follows: For any $p \in X$ and $t \in (0, 1]$, we have

$$d((\Phi_{p,t})_*\mathcal{H}^n)(x) \geq \frac{t\mathrm{sn}_\kappa(t\mathrm{r}_p(x))^{n-1}}{\mathrm{sn}_\kappa(\mathrm{r}_p(x))^{n-1}} d\mathcal{H}^n(x) \tag{2}$$

for any $x \in X$ ($\mathrm{r}_p(x) < \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$), where $(\Phi_{p,t})_*\mathcal{H}^n$ is the push-forward of the measure $\mathcal{H}^n$ by $\Phi_{p,t}$. In Riemannian case, the condition $BG(n, \kappa)$ is equivalent to $\mathrm{Ric} \geq (n-1)\kappa$. This definition reflects one of key properties of the Ricci curvature on Riemannian manifolds, i.e. the infinitesimal volume comprison. There is another notion of volume comparison called $MCP(n, \kappa)$, which was defined by Sturm [17] and Ohta [14]. $MCP(n, \kappa)$ is equivalent to $BG(n, \kappa)$ in the Alexandrov space, see [14, 9].

The infinitesimal version of Bishop-Gromov volume comparison implies the global version of Bishop-Gromov volume comparison, also called relative volume comparison, see [9]. Denote by $B_R(p) := \{x \in X : d(x, p) < R\}$ the geodesic ball of radius R centered at p, by $|B_R(p)| := \mathcal{H}^n(B_R(p))$ the volume of the ball $B_R(p)$ and by $V_r^{n,\kappa}$ the volume of a geodesic ball of radius r in the space form $\Pi^{n,\kappa}$, i.e. the complete simply connected n-dimensional Riemannian manifolds with constant sectional curvature κ.

Theorem 2.1 ([9]). *Let X be an Alexandrov space satisfying $BG(n, \kappa)$, $\kappa \leq 0$. Then for any $p \in X$ and $0 < r < R$, we have*

$$\frac{|B_R(p)|}{|B_r(p)|} \leq \frac{V_R^{n,\kappa}}{V_r^{n,\kappa}} \leq e^{2\sqrt{-(n-1)\kappa}R} \left(\frac{R}{r}\right)^n. \tag{3}$$

Zhang-Zhu [21, 22] introduced a geometric version of lower Ricci curvature bounds on Alexnadrov spaces, denoted by $\mathrm{Ric} \geq -K$ ($K \geq 0$). This definition reflects another important feature of Ricci curvature in Riemannian case, the Bochner formula. For details, we refer to [23]. It was proved [21] that $\mathrm{Ric} \geq -K$ for an n-dimensional Alexandrov space implies Lott-Villani-Sturm's [11, 12, 17] curvature dimension condition $CD(n, K)$ and Kuwae-Shioya's infinitesimal Bishop-Gromov comparison $BG(n, -\frac{K}{n-1})$.

We recall some basic analysis on Alexandrov spaces. For a domain $\Omega \subset X$, we denote by $\mathrm{Lip}(\Omega)$ ($\mathrm{Lip}_0(\Omega)$) the space of (compact supported) Lipschitz functions on Ω. It can be shown that every Lipschitz function is differentiable $\mathcal{H}^n$-almost everywhere and has the bounded gradient (see Cheeger [3]). In this paper, we take $\mathcal{H}^n$ as the background measure

of an n-dimensional Alexandrov space X and implicitly integrate functions over $\mathcal{H}^n$. For a precompact domain $\Omega' \subset\subset X$ and $f \in \mathrm{Lip}(\Omega')$, the $W^{1,2}$ norm of f is defined as

$$||f||^2_{W^{1,2}(\Omega')} = \int_{\Omega'} f^2 + \int_{\Omega'} |\nabla f|^2.$$

The space $W^{1,2}(\Omega')$ (resp. $W_0^{1,2}(\Omega')$) is the completion of $\mathrm{Lip}(\Omega')$ (resp. $\mathrm{Lip}_0(\Omega')$) with respect to the $W^{1,2}$ norm defined above. For a domain $\Omega \subset X$, the local $W^{1,2}$ space of Ω, $W^{1,2}_{\mathrm{loc}}(\Omega)$, consists of functions f with $f|_{\Omega'} \in W^{1,2}(\Omega')$ for any $\Omega' \subset\subset X$. For $f \in L^2(\Omega)$ and $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, we define a linear functional $\mathcal{L}_u$ on Ω (corresponding to Δu in the smooth setting) by

$$\mathcal{L}_u(\phi) = -\int_\Omega \langle \nabla u, \nabla \phi\rangle \quad \text{for } \forall \phi \in \mathrm{Lip}_0(\Omega).$$

In general, $\mathcal{L}_u$ is a signed Radon measure on Ω. We say a function $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ solves the Poisson equation $\mathcal{L}_u = f \cdot \mathcal{H}^n$ on Ω if

$$\mathcal{L}_u(\phi) = \int_\Omega f \cdot \phi \quad \text{for } \forall \phi \in \mathrm{Lip}_0(\Omega).$$

Zhang-Zhu [23] established the following Bochner formula on Alexandrov spaces.

Theorem 2.2 (Zhang-Zhu [23]). *Let Ω be a domain in an n-dimensional Alexandrov space X with* $\mathrm{Ric} \geq -K$. *For an a.e. continuous function $f \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, let $u \in \mathrm{Lip}(\Omega)$ solve the Poisson equation $\mathcal{L}_u = f \cdot \mathcal{H}^n$. Then $|\nabla u|^2 \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and*

$$\frac{1}{2}\mathcal{L}_{|\nabla u|^2} \geq \left(\frac{1}{n} f^2 + \langle \nabla u, \nabla f\rangle - K|\nabla u|^2\right) \cdot \mathcal{H}^n. \tag{4}$$

3. Local uniform Poincaré inequality and Sobolev inequality

The Sobolev inequality is necessary to carry out Moser's iteration. In view of the standard theory for general metric measure spaces, one needs a volume doubling condition and a local uniform Poincaré inequality to prove the local Sobolev inequality. In fact, for an Alexandrov space with $\mathrm{Ric} \geq -K$ there are stronger volume growth properties, i.e. infinitesimal Bishop-Gromov volume comparison $BG(n, -K/(n-1))$ and the global version (3). The standard technique of the change of variables combining with the infinitesimal Bishop-Gromov volume comparison $BG(n, -K/(n-1))$ yields the following Poincaré inequality, see e.g. [6, 16].

Lemma 3.1 (local uniform Poincaré inequality). *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exist $C = C(n)$ such that for $B_R \subset X$ and $u \in W^{1,2}_{loc}(B_R)$,*

$$\int_{B_R} |u - \overline{u}|^2 \leq C e^{C\sqrt{K}R} R^2 \int_{B_R} |\nabla u|^2. \tag{5}$$

As long as the uniform local Poincaré inequality and the Bishop-Gromov volume comparison (3) are available, one can follow the same argument of Lemma 3.5 by setting $A(R) = \sqrt{K}R$ in Munteanu-Wang [13] (see also [5]) to prove the following local uniform Sobolev inequality.

Theorem 3.1 (local uniform Sobolev inequality)**.** *Let X be an n-dimensional Alexandrov space with Ricci curvature bounded from below by $-K$ ($K > 0$). Then there exist two constants $\nu > 2$ and C, both depending only on n, such that for $B_R \subset X$ and $u \in W^{1,2}_{loc}(B_R)$,*

$$\left(\int_{B_R}(u-\bar{u})^{\frac{2\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{K}R)}R^2|B_R|^{-\frac{2}{\nu}}\int_{B_R}|\nabla u|^2, \tag{6}$$

where $\bar{u} = \frac{1}{|B_R|}\int_{B_R} u$. In particular,

$$\left(\int_{B_R}u^{\frac{2\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}} \le e^{C(1+\sqrt{K}R)}R^2|B_R|^{-\frac{2}{\nu}}\int_{B_R}(|\nabla u|^2 + R^{-2}u^2). \tag{7}$$

4. Proof of Theorem 1.1

Without loss of generality, we may assume that u is a positive harmonic function on B_{4R}. It was proved in [15] and [23] that u is locally Lipschitz continuous in B_{4R}, $|\nabla u|$ is lower semi-continuous in B_{4R} and $|\nabla u|^2 \in W^{1,2}_{loc}(B_{4R})$. Denote $v = \log u$. One can easily verify that for $0 \le \eta \in \mathrm{Lip}_0(B_{2R})$,

$$\int_{B_{2R}}\langle\nabla v,\nabla\eta\rangle = \int_{B_{2R}}\eta|\nabla v|^2. \tag{8}$$

Let $f = |\nabla v|^2$. Then $f \in W^{1,2}(B_{2R})\bigcap L^\infty(B_{2R})$. It follows from the Bochner formula (4) and (8) that for $0 \le \eta \in \mathrm{Lip}_0(B_{2R})$,

$$\int_{B_{2R}}\langle\nabla\eta,\nabla f\rangle \le \int_{B_{2R}}\eta\left(2\langle\nabla v,\nabla f\rangle + 2Kf - \frac{2}{n}f^2\right). \tag{9}$$

In fact, by an approximation argument, (9) holds for any $0 \le \eta \in W^{1,2}_0(B_{2R}) \cap L^\infty(B_{2R})$. Let $\eta = \phi^2 f^\beta$, with $\phi \in \mathrm{Lip}_0(B_{2R})$, $0 \le \phi \le 1$ and $\beta \ge 1$. Then η is an admissible test function for (9). Hence we have from (9) that

$$\begin{aligned}&\int_{B_{2R}}\beta\phi^2 f^{\beta-1}|\nabla f|^2 + 2\phi f^\beta\langle\nabla f,\nabla\phi\rangle\\ \le\ &\int_{B_{2R}}\phi^2 f^\beta\left(2\langle\nabla v,\nabla f\rangle + 2Kf - \frac{2}{n}f^2\right).\end{aligned}$$

It follows that

$$\begin{aligned}\frac{4\beta}{(\beta+1)^2}\int_{B_{2R}}\phi^2|\nabla f^{\frac{\beta+1}{2}}|^2 &\leq \frac{4}{\beta+1}\int_{B_{2R}}\phi f^{\frac{\beta+1}{2}}|\nabla\phi||\nabla f^{\frac{\beta+1}{2}}| \\ &\quad +\frac{4}{\beta+1}\int_{B_{2R}}\phi^2 f^{\frac{\beta+2}{2}}|\nabla f^{\frac{\beta+1}{2}}| \\ &\quad -\int_{B_{2R}}\frac{2}{n}\phi^2 f^{\beta+2}+\int_{B_{2R}}2K\phi^2 f^{\beta+1}.\end{aligned}$$

Using Hölder inequality, we obtain

$$\begin{aligned}\int_{B_{2R}}\phi^2|\nabla f^{\frac{\beta+1}{2}}|^2 &\leq C_1\int_{B_{2R}}|\nabla\phi|^2 f^{\beta+1}+C_2\int_{B_{2R}}\phi^2 f^{\beta+2} \\ &\quad -C_3\beta\int_{B_{2R}}\phi^2 f^{\beta+2}+C_4\beta K\int_{B_{2R}}\phi^2 f^{\beta+1}.\end{aligned}$$

We remark that from now on, the constant $C_1, C_2, \cdots$, depend only on n.

For $\beta \geq \frac{2C_2}{C_3}$, we have

$$\begin{aligned}&\int_{B_{2R}}|\nabla(\phi f^{\frac{\beta+1}{2}})|^2+\frac{1}{2}C_3\beta\int_{B_{2R}}\phi^2 f^{\beta+2} \\ &\leq 2C_1\int_{B_{2R}}|\nabla\phi|^2 f^{\beta+1}+C_4\beta K\int_{B_{2R}}\phi^2 f^{\beta+1}.\end{aligned}\tag{10}$$

Using Sobolev inequality (7), we obtain

$$\begin{aligned}\left(\int_{B_{2R}}\phi^{2\chi}f^{(\beta+1)\chi}\right)^{\frac{1}{\chi}} &\leq e^{C_5(1+\sqrt{K}R)}R^2|B_{2R}|^{-\frac{2}{\nu}}\Big(C_6\int_{B_{2R}}|\nabla\phi|^2 f^{\beta+1} \\ &\quad +(C_7\beta K+C_8R^{-2})\int_{B_{2R}}\phi^2 f^{\beta+1}-\beta\int_{B_{2R}}\phi^2 f^{\beta+2}\Big).\end{aligned}\tag{11}$$

where $\chi = \frac{\nu}{\nu-2}$.

We first use (11) to prove the following

Lemma 4.1. *There exists a large positive constant $C = C(n)$ such that for $\beta_0 = C(1+\sqrt{K}R)$ and $\beta_1 = (\beta_0+1)\chi$, we have $f \in L^{\beta_1}(B_{\frac{3}{2}R})$ and*

$$\|f\|_{L^{\beta_1}\left(B_{\frac{3}{2}R}\right)} \leq C_9\frac{(1+\sqrt{K}R)^2}{R^2}|B_{2R}|^{\frac{1}{\beta_1}}.\tag{12}$$

Proof. Let $C_{10} \geq \frac{2C_2}{C_3}$ such that $\beta_0 = C_{10}(1+\sqrt{K}R)$ satisfies (10) and (11). we rewrite (11) for $\beta = \beta_0$ as

$$\left(\int_{B_{2R}} \phi^{2\chi} f^{(\beta_0+1)\chi}\right)^{\frac{1}{\chi}} \leq e^{C_{11}\beta_0}|B_{2R}|^{-\frac{2}{\nu}}\Big(C_6R^2\int_{B_{2R}}|\nabla\phi|^2 f^{\beta_0+1} + C_{12}\beta_0^3\int_{B_{2R}}\phi^2 f^{\beta_0+1} - \beta_0R^2\int_{B_{2R}}\phi^2 f^{\beta_0+2}\Big). \tag{13}$$

We estimate the second term in RHS of (13) as follows,

$$\begin{aligned} C_{12}\beta_0^3\int_{B_{2R}}\phi^2 f^{\beta_0+1} &= C_{12}\beta_0^3\left(\int_{\{f\geq 2C_{12}\beta_0^2R^{-2}\}}\phi^2 f^{\beta_0+1} + \int_{\{f<2C_{12}\beta_0^2R^{-2}\}}\phi^2 f^{\beta_0+1}\right) \\ &\leq \frac{1}{2}\beta_0R^2\int_{B_{2R}}\phi^2 f^{\beta_0+2} + C_{13}^{\beta_0+1}\beta_0^3\left(\frac{\beta_0}{R}\right)^{2(\beta_0+1)}|B_{2R}|. \end{aligned} \tag{14}$$

Let us set $\phi = \psi^{\beta_0+2}$ with $\psi \in \mathrm{Lip}_0(B_{2R})$ satisfying

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \text{ in } B_{\frac{3}{2}R},\ |\nabla\psi| \leq \frac{C_{14}}{R}.$$

Hence $R^2|\nabla\phi|^2 \leq C_{15}\beta_0^2\phi^{\frac{2(\beta_0+1)}{\beta_0+2}}$. Then by Hölder and Young inequalities, the first term in RHS of (13) can be estimated as follows:

$$\begin{aligned} C_6R^2\int_{B_{2R}}|\nabla\phi|^2 f^{\beta_0+1} &\leq C_{16}\beta_0^2\int_{B_{2R}}\phi^{\frac{2(\beta_0+1)}{\beta_0+2}} f^{\beta_0+1} \\ &\leq C_{16}\beta_0^2\left(\int_{B_{2R}}\phi^2 f^{\beta_0+2}\right)^{\frac{\beta_0+1}{\beta_0+2}}|B_{2R}|^{\frac{1}{\beta_0+2}} \\ &\leq \frac{1}{2}\beta_0R^2\int_{B_{2R}}\phi^2 f^{\beta_0+2} + C_{17}\beta_0^{\beta_0+3}R^{-2(\beta_0+1)}|B_{2R}|. \end{aligned} \tag{15}$$

Substituting the estimates (14) and (15) into (13), we obtain

$$\left(\int_{B_{2R}}\phi^{2\chi} f^{(\beta_0+1)\chi}\right)^{\frac{1}{\chi}} \leq 2e^{C_{11}\beta_0}C_{13}^{\beta_0+1}\beta_0^3\left(\frac{\beta_0}{R}\right)^{2(\beta_0+1)}|B_{2R}|^{1-\frac{2}{\nu}}.$$

Taking the (β_0+1)-th root on both sides, we get

$$||f||_{L^{\beta_1}(B_{\frac{3}{2}R})} \leq C_{18}\left(\frac{\beta_0}{R}\right)^2|B_{2R}|^{\frac{1}{\beta_1}}.$$

□

Now we start from (11) and use the standard Moser iteration to prove Theorem 1.1. Let $R_k = R + \frac{R}{2^k}$, $\phi_k \in \mathrm{Lip}_0(B_{R_k})$ satisfies

$$0 \leq \phi_k \leq 1, \quad \phi_k \equiv 1 \text{ in } B_{R_{k+1}}, \quad |\nabla\phi_k| \leq C\frac{2^{k+1}}{R}.$$

Let β_0, β_1 be the numbers in Lemma 4.1 and $\beta_{k+1} = \beta_k \chi$ for $k \geq 1$, one can deduce from (11) with $\beta + 1 = \beta_k$ and $\phi = \phi_k$ that (we have dropped the last term in the RHS of (11) since it is negative)

$$\|f\|_{L^{\beta_{k+1}}(B_{R_{k+1}})} \leq e^{C_{19}\frac{\beta_0}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\frac{1}{\beta_k}} (4^k + \beta_0^2 \beta_k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_k}(B_{R_k})}.$$

Hence by iteration we get

$$\|f\|_{L^\infty(B_R)} \leq e^{C_{19}\beta_0 \sum_k \frac{1}{\beta_k}} |B_{2R}|^{-\frac{2}{\nu}\sum_k \frac{1}{\beta_k}} \prod_k (4^k + 2\beta_0^3 \chi^k)^{\frac{1}{\beta_k}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}.$$

Since $\sum_k \frac{1}{\beta_k} = \frac{\nu}{2}\frac{1}{\beta_1}$ and $\sum_k \frac{k}{\beta_k}$ converges, we have

$$\begin{aligned} \|f\|_{L^\infty(B_R)} &\leq C_{20} e^{C_{21}\frac{\beta_0}{\beta_1}} \beta_0^{\frac{3\nu}{2}\frac{1}{\beta_1}} |B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})} \\ &\leq C_{22} |B_{2R}|^{-\frac{1}{\beta_1}} \|f\|_{L^{\beta_1}(B_{\frac{3}{2}R})}. \end{aligned}$$

Using Lemma 4.1, we conclude

$$\|f\|_{L^\infty(B_R)} \leq C(n) \frac{(1 + \sqrt{K}R)^2}{R^2},$$

which implies

$$\|\nabla \log u\|_{L^\infty(B_R)} \leq C(n) \frac{1 + \sqrt{K}R}{R}.$$

This proves the Theorem 1.1.

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